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TERM STRUCTURE OF CREDIT SPREADS WITH AFFINE PROCESSES

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Concluded in:

ABSTRACT

A brief review and a comparison between Structural and Intensity mode is presented in the first section with an argument in favor of Intensity models based on the quality of information available to market participants.

Next, an Intensity model with an affine (constant plus linear) parametrization of the intensity parameter driven a by a set of latent variables is formulated for the Euro Credit Default Swap (CDS) curve. Latent variables in the intensity parameter are assumed to follow uncorrelated CIR processes. Furthermore the model parameters, the latent variable processes and implicit risk-neutral default probabilities are estimated with an application of a Linearized Kalman Filter approach and a Likelihood maximization algorithm.

A conclusion is reached that a model with two latent variables is able to account for 95%, 89%, 98% and 99% of the variations in 3, 5, 7 and 10-year maturities of the CDS curve.

Keywords: credit risk, affine, intensity models, CIR, term-structure, kalman filter.

JEL Classification: G13, G22.

ESTRUTURA TEMPORAL DE SPREADS DE CRÉDITO COM PROCESSOS AFFINE

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Provas Concluidas em:

RESUMO

Um breve resumo e uma comparação entre os modelo Estruturais e modelos de forma Intensiva é apresentado na primeira secção do trabalho, com um argumento a favor dos modelos de forma Intensiva baseado na qualidade de informação disponível aos participantes do mercado.

De seguida, é formulado um modelo de forma Intensiva para a a curva "Credit Default Swap (CDS)" da zona Euro com a parametrização "affine" (constante mais linear) do parâmetro de intensidade que por sua vez é movido por um conjunto de variáveis latentes. É assumido que as variáveis latentes seguem processos CIR não correlacionados. Os parâmetros do modelo, processos das variáveis latentes e probabilidade de incumprimentos risco-neutrais implícitas são estimados com a aplicação da versão Linearizada do Filtro Kalman em conjunto com o algoritmo de Maxima Verosimilhança.

Por fim conclui-se que o modelo com duas variáveis latentes é capaz de explicar 95%, 89%, 98% e 99% das variações nas maturidades de 3,5, 7 e 10 anos da curva CDS.

Palavras Chave: risco de crédito, affine, modelos de forma intensiva, CIR, estrutura temporal, filtro kalman.

Classificação JEL: G13, G22.

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1 Introduction

The purpose of this paper is to study the forces responsible for movements in credit spreads at different maturities. A credit spread is usually perceived as the premium demanded by the investors to hold risky (subject to default) assets over risk-free counterparts. Examples of these assets are corporate bonds that have credit risk embedded due to the possibility of a company's default. On the other hand examples of risk-free assets are bonds issued by governments of developed countries carrying zero[1] credit risk due to the general belief that it is impossible for example the United States government to default on its obligations. A common misleading interpretation emerges: The difference in return between risky and risk-free bonds is the credit spread. This interpretation is incorrect because there are two additional components in this difference besides credit spread: liquidity premium and tax premium. The presence of these two components imply that bonds are not an ideal choice for the study of credit spreads.

Alternative financial instruments are available on the markets that bear a better reflection of credit spreads. One that focuses directly on credit risk and establishes a price for it in a market environment is a Credit Default Swap (CDS) contract. CDS contracts are credit derivatives with a standardized specification dictated by the ISDA[2] and are comprised of three main components: a risky underlying asset; the contracting party normally holding this asset and an insuring party. The risky asset is usually a senior, unsecured bond emitted by a corporation subject to default risk. The contracting party holding this asset enters into such a contract with the view to offset the potential default risk of the corporation in return for a semiannual premium payment to the insuring party. Thus a CDS contract takes the form of a standard insurance policy where the insured capital is the risky asset and where the premium prices directly reflects the risk involved.

CDS contracts apart from having an appealing definition for the credit risk subject, also benefit from being traded on organized Over the Counter Markets with fixed maturities from 1 to 10 years. Availability of a multidimensional dataset lets one study the time-series and cross-section properties of the variable. Models that take into consideration these two dimensions are commonly referred to as *term structure* models.

The paper proceeds as follows: In section 2 structural and intensity credit models are compared. Section 3 provides a description of the CDS timeseries properties and in section 4 a formulation of an intensity model is presented. In section 5 the model estimation methodology is discussed and in section 6 results are presented. I conclude with section 7.

[1] For the sace of this dicussion I assume that government bonds are risk-free. Government bonds of developed countries with a solid politic system, stable economy and sound financial markets are usually attributed a AAA rating by the rating agencies which is although not risk-free, but very close to.

[2] International Swaps and Derivative Association (http://www.isda.org)

2 Credit Models

In the theory of credit risk one is generally limited to a choice between two different modeling approaches: The classic *Structural* models, or the more recent *Intensity* models. These are generally perceived as rivals, until the recent publication of two papers by Duffie & Lando (2001) and Çetin *et al.* (2004) demonstrating that the two models, although based on distinct assumptions are actually of the same kind if an additional relatively weak assumption is added to both of them. This vital assumption linking both models into one is that the market participants are less informed than the managers. A distinction between information sets implies difference in perception of a company's health by two sets of agents, the ones participating in the market and the ones controlling the company.

In order to get a better grasp of the two modeling approaches, in the next two subsections I will briefly review both of them and provide an intuition on under what circumstances one should be chosen over the other.

2.1 Structural Credit Models

Since the early 70's with the publication of two influential papers on pricing of defaultable contingent claims by Merton (1974) and Black & Cox (1976), their ideas became standard for credit risk modelling. Almost exclusively all subsequent thirty years of research were based on their fundamental idea that a firm's value could be modelled as a standard, continuous diffusion process with default occurring when a firm's value drops to a level where repayment of outstanding debt can not longer be warranted. A number of commercial packages, allegedly predicting the risk of a firm's default have been developed, such as Moody's KMV or JPMorgan - CreditGrades [3]. The term *Structural* is at times used to classify these models, as explicitly defining the structure of how a firm's value evolves throughout time.

Consider a firm's value to be well represented by a diffusion process where μ is the firm's asset rate of return, σ asset volatility and W_t a standard Brownian motion

$$dV_t = \mu V_t dt + \sigma V_t dW_t \tag{1}$$

This equation intuitively tells us that a firm's value is expected to fluctuate randomly around its long-term growth path defined by the variable μ. Fluctuations are not deterministic and thus do not permit to infer exactly on where a firm's value will be at any period in time, but it provides

[3] Documentation on Moody's KMV and CreditGrades models can be found by the following to links respectively: http://www.moodyskmv.com/research/index.html
http://www.creditgrades.com/resources/resources

enough information to construct a probability distribution on the likelihood of being above, or below a certain level at any future date.

Lets now consider that for a firm to stay alive its asset value must at all times remain above a certain barrier K. This assumption is quite reasonable as many firms in the market carry a significant amount of debt, and if their asset value drops below the value of debt, the firm technically goes into default. Many times it is convenient to have an estimate of the probability that a firm will not default before some time τ, as in the case when a decision is made on whether to lend funds to a firm and under what conditions.

Formally the probability of survival $\mathbb{Q}$ in structural models depends on a firm's internal parameters μ, σ, current asset value V_0 and the default barrier K, as

$$\mathbb{Q}(\tau > T) = f(V_0, K, \mu, \sigma, T) \tag{2}$$

where the implicit function $f(\cdot)$ can vary according to the definition of randomness in the asset process. If dW is assumed to follow a normal distribution, as is usually done, then function $f(\cdot)$ will be something similar to the well-known Black-Scholes equation.

2.2 Intensity Credit Models

Intensity models were first introduced by Jarrow & Turnbull (1995), Jarrow *et al.* (1997) and extended in papers such as Duffie *et al.* (1996), Lando (1998), Duffie & Singleton (1999), Duffee (1999). These models bear significant differences to structural models. As opposed to structural models, intensity models are silent about why default occurs. They do not make assumptions on the evolution of firm's asset value. Moreover, capital structure, volatility, leverage ratio and expected return play no direct role in the definition of the survival probability. In intensity models survival probability is calculated on the basis of a binary outcome of a point process. A firm defaults if the underlying process assumes a value of 1, implying that the probability of surviving is equivalent to the probability of a point process not jumping from 0 to 1.

The term *intensity* comes from the definition of the underlying point processes. A Poisson process is one example where the intensity Λ is the only variable controlling the probability of a jump. In fact Poisson processes have much in common with the general specification of intensity models. Consider a standard Poisson process[4] with a known intensity, the probability that the

[4] From statistics, a homogeneous (time independent) Poisson process, with terminology adapted to credit risk, states that the probability of n defaults within a fixed time interval is controlled by an intensity Λ through the relation

$$P(n = k) = \frac{e^{-\Lambda}\Lambda^k}{k!}$$

process will not jump during a certain time interval depends only on the intensity for the same time interval through the relation

$$P(n = 0) = e^{-\Lambda} \tag{3}$$

Extending this logic to the construction of intensity models, we can define the probability of survival past time T as

$$\mathbb{Q}(\tau > T) = e^{-\Lambda} \tag{4}$$

This last equation is the main building block for intensity models, extremely simplistic in its construction, but sufficiently versatile to accommodate a vast variety of parametrization. In fact, as we will see later, the intensity parameter Λ is not required to remain constant, or to be deterministic. The intensity variable can be allowed to take on alternative functional forms, be affected by external macroeconomic and internal, firm specific variables of deterministic or stochastic nature.

2.3 Equivalence between Intensity and Structural Models

As in Jarrow & Protter (2004). Structural models assume complete knowledge of a very detailed information set (up to the stochastic process of the asset price). This information is normally thought to be available to managers. With an extremely refined information set, up to the point of knowing the value of assets at each point in time implies that a firm's default is predictable, but this is not necessary the case. In contrast, intensity models assume knowledge of a less detailed information set (up to the knowledge of the default probability). This information set is normally though to be available to the market.

Duffie & Lando (2001) and Çetin *et al.* (2004) demonstrate that structural and intensity models are equivalent by assuming that the asset diffusion process in the structural models is not perfectly observable. An interpretation is that accounting reports and press releases either purposefully or inadequately add extra noise that obscures the market's knowledge of the firm's asset value.

Both papers reach the same, crucial conclusion that brought intensity models greater attention and acceptance from academics and practitioners: *If the asset value is not perfectly observable then the structural models are just a special case of more general intensity models.* Intuitively, if the asset value is not perfectly observable then the market has no way of knowing the exact function in equation 1 and mathematically by adding noise to this function makes it reducible to the one of intensity models in equation 4.

When choosing a model one must take into consideration two main factors: the model's ability to mirror empiric stylized facts and to admit a viable specification for econometric estimation. More flexible intensity models are able to better adhere to these two requisitions and in addition, as argued above, have the advantage of nesting a structural model within. Since the publication of seminal papers on intensity models, they have been slowly becoming the models of choice especially in term-structure modeling problems. Recent examples of their applications to CDS premia are those of Schneider *et al.* (2007) and Pan & Singleton (2007).

3 Data Description

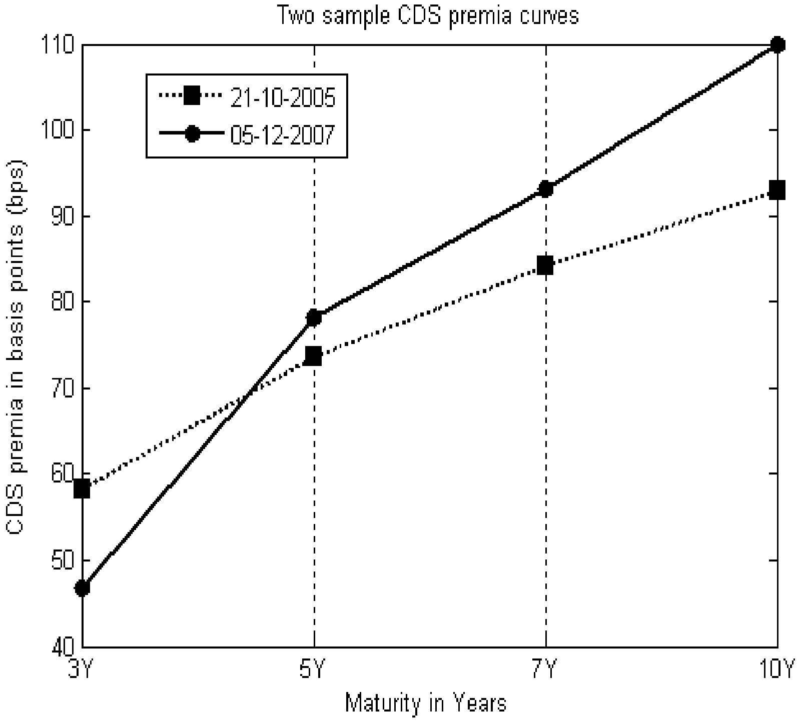
According to Reyngold *et al.* (2007), CDS prices are fairly pure indicators of credit risk because their structure separates the credit risk component from other asset risks and premiums, such as: Interest rate risk, currency risk and tax premium normally present in bonds. In addition they are light instruments in the sense that one does not need to fund an entire bond position, for example, to have essentially the identical credit risk exposure. Finally CDS pricing has become liquid with standardised ISDA contracts and exponentially growing markets. Total market size of CDS (notional outstanding) was 17 trillion USD by broker estimates in April 2006.

In the empirical application I will use a CDS iTraxx index of High Volatility European companies taken from Reuters Xtra 3000 and consider it as a good overall credit risk representation in the euro area. This specific series was chosen due to its relatively extensive timeseries, greater number of maturities if compared to other iTraxx series and more liquidity if compared to contracts on individual counterparties. Daily iTraxx HiVol series for 3, 5, 7 and 10 year maturities, spanning from 21-10-2005 to 5-12-2007, with a total of 537 data points were extracted. The series' descriptive statistics are very similar to what is expected from any financial series: a skewed distribution with a large kurtosis, indicating non-normality. Two sample CDS curves are presented in Figure 1.

3.1 Sample Distribution Statistics

When looking at a time series distribution it is usually sufficient to analyse the first four moments to get a picture of its characteristics: mean, variance, skewness and kurtosis. These are calculated for first differences of the original series in order to remove any possible trends in data: $\hat{S}_t = S_t - S_{t-1}$

Figure 1: Two CDS curve examples, one from the beginning and the other from the end of the sample. On the vertical axes the CDS premia is quoted in basis points (1 bps = 0.01%).



and presented in a table below for the available range of maturities

Table 1: Descriptive Statistics of CDS premia in Differences[5]

	3Y	5Y	7Y	10Y
Standard Deviation	0.00015	0.00021	0.00022	0.00021
Skewness	0.2647	-0.1278	-0.7206	-0.4733
Kurtosis	10.5127	13.8708	16.7872	15.3054

A first observation can be made, shorter maturity contracts tend to exhibit higher volatilities, especially the 3 year contracts with more than 3 times the volatility of the 10 year contract. This can be an indication that shorter maturity contracts respond more to short-term market turmoils while the longer maturities are more affected by long-term economic trends that tend to be less volatile.

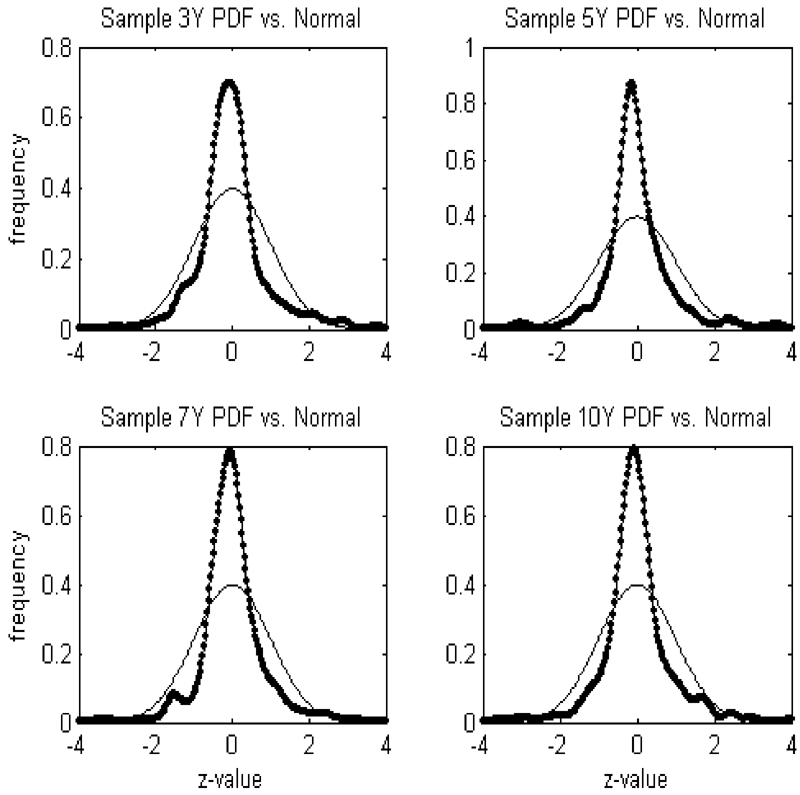
Skewness gives an indication on the sample distribution tilt, leftward (negative value) or rightward (positive value) and the size of this tilt. It is surprising to see that only the 3 year maturity exhibits a positive skewness indicating mostly positive movements. The fourth moment, Kurtosis,

[5] The series mean value is omitted due to being statistically indiferent from zero

gives an indication on the concentration of values around the distribution mean, as well as the tail size. A time-series with a tight concentration of data points around the distribution mean, and a significant number of outliers is bound to exhibit a large kurtosis. CDS time-series is no exception with kurtosis above 10 across all maturities (in comparison to a kurtosis of 3 in a normal distribution).

Figure 2 visually confirms the analysed descriptive statistics and motivates the argument against a single, Gaussian model to capture the full distribution spectrum of CDS premia, such as a structural model.

Figure 2: Comparison between the diferenciated CDS premia series and the normal distribution.



4 Intensity Model for CDS Premia

CDS contracts are credit derivatives that give the buyer protection against default of the underlying asset and issuer, the exposure to this credit risk. Individual CDS contracts on an underlying asset are constructed in such a form that in case of a default, the selling party is obliged to reimburse the buying party the full covered notional amount. In return, up to the date of the default the buying party agrees to make quarterly premium payments for the benefit of this protection, in a much similar way to a common insurance policy. In a complete market where no arbitrage opportunities may exist, equilibrium is attained when the expected sum of all the premium payments are equal to expected pay-out in case of a default.

I will denote A_0 as the expected sum of quarterly premiums payments by the buying party over the life of the contract. Where c is the constant premium, T the term of the contract, r the risk free interest rate[6] and $1_{\tau>T}$ an indicator function taking on value 1 if no default occurs prior to T and 0 otherwise. Under the risk neutral measure $\mathbb{Q}$, the expected current value of all the premiums is a relation between the risk free interest rate and the indicator function implying premium payments up to the date of the default

$$A_0 = E^{\mathbb{Q}}\left[c\sum_{j=1}^{4T} e^{-r\left(\frac{j}{4}\right)} 1_{\tau>\frac{j}{4}}\right] \tag{5}$$

In the world where interest rates are deterministic one can work through the expectation and arrive at a probability for the indicator function, being the only unknown parameter. The equation above can thus be simplified to take on a more convenient form

$$A_0 = c\sum_{j=1}^{4T} e^{-r\left(\frac{j}{4}\right)} \mathbb{Q}\left(\tau > \frac{j}{4}\right) \tag{6}$$

where as has been seen before, $\mathbb{Q}\left(\tau > \frac{j}{4}\right)$ is the probability of survival past $\frac{j}{4}$ corresponding to the periodic premium payment date.

Further I will denote B_0 as the expected present value of the pay-out in the case of a default, with all other parameters as before

$$B_0 = E^{\mathbb{Q}}\left[e^{-rT}\left[1 - 1_{\tau>T}\right]\right] \tag{7}$$

It is trivial to see that if $1_{\tau>T}$ indicates survival past time T, then $1 - 1_{\tau>T}$ indicates the contrary, default prior to time T. Working through the expectation, and as before with deterministic interest rates, the above equation simplifies to

$$B_0 = e^{-rT}\left[1 - \mathbb{Q}\left(\tau > T\right)\right] \tag{8}$$

Furthermore, in an arbitrage free market, the equality relation between A_0 and B_0 must hold. By equaling the two and solving for the premium parameter I arrive at an equation relating the

[6] For the interest rate I will use the German zero-coupon Bunds at available maturities and fill missing data points through linear interpolation.

CDS premia to the probabilities of survival

$$c = \frac{e^{-rT}\left[1 - \mathbb{Q}\left(\tau > T\right)\right]}{\sum_{j=1}^{4T} e^{-r\left(\frac{j}{4}\right)} \mathbb{Q}\left(\tau > \frac{j}{4}\right)} \tag{9}$$

From here onward one can either estimate the premium market price if the survival probabilities are known, or estimate the probabilities of survival if market premiums are known. Information regarding probabilities of survival are regularly published by rating agencies, thus letting one estimate CDS premia by a simple application of the above equation. The objective of this paper is however, related to the latter problem, estimating not only the probabilities of survival, but also extracting the forces that move them.

4.1 Intensity Representation of the Survival Probability

The next question is how to give some explicit form to the probability of survival. As it has been seen in the introductory section, intensity models are built on point processes where the probability of survival is defined as the probability of no jumps in the point process within a certain time interval. The probability of survival past time T is given by equation 4 which could be back substituted into equation 9 to get rid of the probability parameter, but this would still leave the model depending on the intensity variable, Λ, with up to now unknown form. The intensity variable could be left constant and estimate likewise, however significantly constraining the model dynamics. An alternative approach is to let the intensity take on a stochastic form, by which adding richer dynamics to the model and letting the probability of survival vary through time.

4.2 Stochastic Affine Intensity

Affine intensity as will be described here and affine models in general are characterized by all the variables having a constant plus linear relation between themselves. What makes affine models more attractable when compared to other functional forms is their direct interpretation of parameters and maybe most important of all, they generally admit closed form solutions. Affine models were first introduced and popularized by Duffie & Kan (1996) with the publication of a seminal paper on interest rate yield-curve modeling. Their initial approach was later extended by Duffie *et al.* (2000) to show that every exponential affine jump-diffusion model admits a closed-form solution and includes a wide array of yield-curve models as special cases, such as the Vasicek (1977) and the Cox *et al.* (1985) model.

Returning to the survival probability, a stochastic version is attained when the intensity follows some non-deterministic process and therefore is no longer constant. Departure from constant intensity implies that the survival probability is no longer constant across its time horizon and must be evaluated over the full trajectory of the intensity variable. In mathematical terms, a stochastic survival probability past time T is

$$\mathbb{Q}(\tau > T) = E^{\mathbb{Q}}\left[e^{-\int_0^T \Lambda(s)ds}\right] \tag{10}$$

The probability is affine in construction if the intensity parameter is a linear function of some underlying latent variables (I will use the terms variables and factors interchangeably throughout this paper). Latent variables, X, take their name from their unobservable nature, similar to principal components in a principal component analysis and in general do not carry any obvious economic interpretation[7]. Formally an affine intensity takes the form

$$\Lambda(X_t) = \lambda_0 + \lambda_1 X_t \tag{11}$$

where there is no constrain on the number of latent variables entering the above equation. In a univariate case X_t is a just a single variable, whereas in a multivariable case X_t and λ_1 can be n-dimensional vectors.

However for a model to be classified as affine it is insufficient for the intensity to be a linear function of latent variables, as the stochastic process governing each latent variable also has to be of affine form[8]. In an Itō representation of the stochastic process for latent variables X_t

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t \tag{12}$$

where $\mu(X_t)$ and $\sigma(X_t)$ are implicit vectors in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ controlling the deterministic and variable parts accordingly, and W_t a standard Brownian motion. The model is affine if both the deterministic and variable parts have a linear plus constant dependence on each other

$$\begin{aligned} \mu(X_t) &= \mu_0 + \mu_1 X_t \\ \sigma(X_t) &= \sqrt{\sigma_0 + \sigma_1 X_t} \end{aligned} \tag{13}$$

[7] In the field of interest rate term structure modeling, attempts have been made to label latent factors with intuitive names such as *level, slope* and *curvature*. For further discussion consult Cristensen *et al.* (2007).

[8] For a technical reference on the necessary and sufficient conditions of affine models, as well as a mathematical demonstraiton of most of its characteristics, Duffie *et al.* (2003) is a standard reference.

4.3 CIR Process as a Special Case

Affine models are malleable enough in the definition of the latent variable diffusion process to encompass a wide variety of alternative specifications, only constrained by the technical conditions as explained in Dai & Singleton (2000). One of these examples is the Cox *et al.* (1985) process, widely referred to as the CIR process initially devised for the modeling of interest rates. A CIR diffusion process takes the form

$$dX_t = k\left(\theta - X_t\right)dt + \sigma\sqrt{X_t}dW_t \tag{14}$$

with constant parameters k, θ and σ. Parameter θ has the interpretation of the long-run mean, k the speed of convergence to the long-run mean and σ volatility. This process is affine, as its parameters map directly to the ones in equation 13, $k\theta = \mu_0$, $k = \mu_1$, $\sigma = \sigma_1^2$.

CIR and Gaussian processes are the best known examples of affine diffusions. The two classes differ with respect to their assumptions about the variance parameter. Gaussian processes have a constant variance while CIR processes introduce conditional heteroskedasticity by allowing the variance to depend on the state variable which as noted by Bates (1996) and Heston (1993) is an important ingredient to account for skewness and large kurtosis in financial timeseries, as the ones presented in section 3.1.

4.4 Loss Given Default (LGD)

Central to the pricing of any insurance policy or a derivative on an asset subject to default is Loss Given Default, or simply LGD. By omitting loss given default, one would tend to overprice the premiums by implicitly assuming that the insurer is subject to bear the full loss of asset value at default, which is not generally the case, as recoveries tend to vary anywhere from 20% in the services sectors to 80% in capital intensive sectors[9].

There are two alternative ways to account for loss given default in the model. The simpler one, which I will use in this paper, usually referred to as *Recovery of Market Value* and the second one, *Recovery of Face Value*. Based on a simulation study, Duffie & Singleton (1999) provide an empiric comparison between the two methods and their main findings suggest that distinction between the two is negligible to a few basis points, thus supporting my choice of the simpler specification.

In the Recovery of Market value[10] formulation, loss given default parameter l enters the model

[9] A study by Altman & Kishore (1996) illustrates that loss give default tends to vary significantly between sectors and provides an estimate for this variable per industry sector on the basis of historic evidence.

[10] Note that Recovery refers to the fraction recovered in the case of a default, while Loss Given Default (LGD) refers to the fraction lost in the case of default. There should be no ambiguity between the two, as Recovery is $1-l$, where l is the loss.

through a multiplicative relation with the intensity and affects only the pay-out part in the case of a default (B_0). In a stochastic affine setting the expected pay-out at default with a non-zero recovery now becomes

$$B_0 = e^{-rT}\left[1-\tilde{\mathbb{Q}}(\tau > T)\right] \tag{15}$$

$$\tilde{\mathbb{Q}}(\tau > T) = E^{\mathbb{Q}}\left[e^{-\int_0^T l\Lambda(s)ds}\right] \tag{16}$$

The intuition behind multiplying the intensity by the loss given default is as follows. Decreases in the intensity translate into increases in the probability of survival which in turn translate into decreases in the premium prices that the insurer is willing to accept under the view that not all of the asset value will be lost in a case of a default. And as l is constrained in the domain $[0,1]$, including it in the model effectively increases the probability of survival by artificially lowering the intensity parameter[11].

4.5 Model Solution

Probably the greatest difficulty in finding a solution to a term structure model, as the one in this paper, is getting rid of expectations and evaluating the integral in the exponential function. Fortunately enough, we can make use of the Feynman-Kac approach by reducing the expectation to a partial differential equation that for an affine configuration always admits a solution of the form $\mathbb{Q}(\tau > T) = e^{\alpha(T)+\beta(T)X}$ where $\alpha(T)$ and $\beta(T)$ solve a partial differential equation. The approach itself is outlined in Appendix 8.1

Explicitly assuming a CIR process for latent variables has a further advantage that the solution has been demonstrated by Cox *et al.* (1985) and takes the form[12]

$$\mathbb{Q}(\tau > T) = E^{\mathbb{Q}}\left[e^{-\int_0^T \Lambda(s)ds}\right] = \alpha(T)e^{-\beta(T)X} \tag{17}$$

where $\alpha(T)$ and $\beta(T)$ are known explicitly[13]

[11] A formal demonstration of how the Recovery of Market Value is introduced into affine intensity models can be found in Duffie (2005).

[12] Note that by assuming a CIR process for latent variables I inevitably simplify equation 11 by setting $\lambda_0 = 0$ and $\lambda_1 = 1$.

[13] While the presented solution is for a univariate case, extentions to include aditional latent variables is straight forward. For two latent variables entering the model, the solution becomes $\mathbb{Q}(\tau > T) = \alpha_1(T)\,\alpha_2(T)\,e^{-\beta_1(T)X_1-\beta_2(T)X_2}$

$$\alpha(T) = \left[\frac{2\gamma e^{(k+q+\gamma)\frac{T}{2}}}{(\gamma+k+q)\left(e^{\gamma T}-1\right)+2\gamma}\right]^{\frac{2k\theta}{\sigma^2}} \tag{18}$$

$$\beta(T) = \frac{2\left(e^{\gamma T}-1\right)}{(\gamma+k+q)\left(e^{\gamma T}-1\right)+2\gamma} \tag{19}$$

$$\gamma = \left((k+q)^2+2\sigma^2\right)^{\frac{1}{2}} \tag{20}$$

The parameters k, θ and σ are from the CIR diffusion process in equation 14 and the variable q is the market risk premium. Positive risk premiums arise for $q < 0$.

5 Model Estimation

With the use of a principal component analysis I conclude that roughly 91% of the movements in CDS premia across all four maturities can be explained by one latent variable and around 95% by two (refer to Appendix 8.4 for details). Based on these statistical facts I estimate two sets of models, the first one with a single latent variable, and another one with two latent variables.

For the estimation method, I use a discrete Kalman Filter approach. As discussed in Chen & Scott (2003), from available alternatives, namely General Method of Moments (GMM) and Simulated Maximum Likelihood (SML), Kalman Filter is usually the preferred choice, as it generally provides lower standard errors for parameter estimates than GMM and is significantly less time consuming than SML. The main disadvantage of the Kalman Filter approach is that it implicitly assumes normality for the estimation error distribution, sometimes difficult to justify in applications to financial time series[14].

Both one-factor and two-factor models share the same set of equations, only varying in the number of parameters, thus without loss of generality I present the general model and then provide estimate results for each one of them.

For the application of the Kalman Filter, the model and the processes governing the evolution of latent variables need to be transformed into a compatible set of equations. The first equation is usually referred to as the *measurement equation,* linking latent variables to observable CDS premia, and the second equation as *transition equation,* defining the value for latent variables at each point in time. The measurement and transition equations to be used in estimation are presented below

[14] A brief review with additional references to alternative estimation methodologies, their advantages and disadvantages is provided in Chen & Scott (2003).

respectively

$$
\begin{aligned}
c &= \frac{e^{-rT}\left[1-\tilde{\alpha}(T)e^{-\tilde{\beta}(T)X}\right]}{\sum_{j=1}^{4T} e^{-r\left(\frac{j}{4}\right)}\alpha(\frac{j}{4})e^{-\beta\left(\frac{j}{4}\right)X}} + R_t & (21)\\
X_t &= \theta\left(1-e^{-k}\right) + e^{-k}X_{t-1} + F_t & (22)
\end{aligned}
$$

The measurement equation is obtained by substituting the probability of default from equation 17 by the function for CDS premia in equation 9 and including an additional error term R_t to account for discrepancies between observed and model predicted values. At this point it is convenient to note that a Kalman Filter is devised to only handle linear functional forms for the measurement equation, while the one presented above is clearly a non-linear one. Slightly differing from the work of Duffee (1999), where the model was linearized around each observation, I linearize the model only once at the beginning of estimation around the long-term mean θ of the latent variable, thus gaining significant improvement in the speed of estimation from roughly two days to a few minutes[15].

The transition equation is a discrete solution to the continuous version of the CIR process in equation 14. As in the continuous version, θ is the long-run mean, k is the speed of convergence to the long-run mean and F_t is the random part of the equation that assumes a somewhat more complicated form[16]. It has been noted by a number of authors including Cox *et al.* (1985), Duffee (1999), Chen & Scott (2003) and Duan & Simonato (1999) that a central problem in estimating CIR processes is that non-negativity of its variance is at times difficult to assure due to the way the latent variable affects the variable part F_t of the process. Negative values in the latent variable lead to negative volatilities. To remedy this problem, similarly to Duffee (1999) I will impose a zero barrier for the latent variables. Alternative solutions for negative variance are discussed in Appendix 8.3.1.

[15] Refer to Appendix 8.3 for a discussion on alternative linearization techniques.

[16] As discussed in the original paper on CIR processes Cox *et al.* (1985), in a diffusion process with square-root volatility the transition distribution is a non central chi-squared, rather than normal as in a gaussian diffusion process. This difference in volatility slightly alters the usual representation of the variable term in the transition equation to

$$F_t = \frac{\sigma^2\left[X_{t-1}\left(e^{-k}-e^{-2k}\right)+\frac{\theta}{2}\left(1-e^{-k}\right)^2\right]}{k}$$

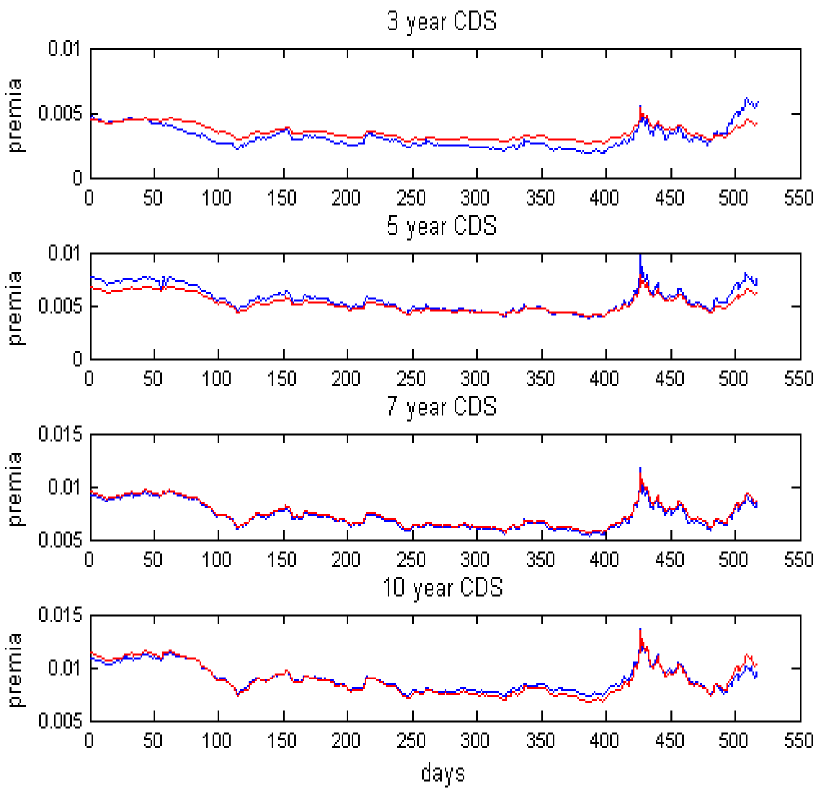
6 Estimation Results

6.1 One-Factor Intensity Model

The results in this section were generated by estimating the model in equation 21 with one latent variable on four CDS maturities with a timeseries comprising 537 daily observations. As expected, the ability of the one-factor model to explain time-series movements at the shortest and the longest maturities is quite limited, although showing some encouraging results for the 5 and 7 year maturities.

Figure 3 illustrates the fit of the model at various maturities. The corresponding R^2 statistic for each of the four maturities is 72%, 81%, 98% and 97% respectively.

Figure 3: Observed and estimated CDS premia for several maturities. The blue line representes the observed values and the red the estimated values.



From the observed results it seems that a single latent variable is not sufficient to fully describe all the datapoints on the CDS curve, especially the ones at lower maturities. From literature it is known that the structure of a yield curve, up to a great accuracy, can be described in terms of its *level, slope* and *curvature*. Static models of Nelson & Siegel (1987) and Svensson (1994) and their dynamic counterparts studied by Cristensen *et al.* (2007) argue that two factor models are able to successfully account for the level and slope of the curve, while a third factor is generally required to fully model its curvature. Thus one can expect that by adding an extra factor into the model, the fit should be improved.

The parameter estimation results are presented in Table 2 with standard errors in parenthesis[17].

Table 2: Parameter estimates for one-factor model

Parameters	Value
k	$5.5E-7$ $(3.4E-4)$
θ	0.0159 $(1.8E-4)$
σ	0.0297 (0.0054)
q	-0.367 (0.0026)

All parameters, except k, are statistically significant. The parameter k being statistically insignificant with a t-statistic of 0.001 implies that the process does not tend to revert to its long-run mean but rather fluctuate like a random walk. Concerning the parameter q, following the argument of Cox *et al.* (1985) negative values for q correspond to positive risk premia. The resulting process for the latent variable is presented in Figure 4.

An interesting note can be made about the volatility parameter of the process σ. Given that the model is estimated with a single factor, its movements should closely resemble the movements in CDS premia which is exactly the case, though with much greater volatility[18]. This is illustrated in Figure 5

6.2 Two-Factor Intensity Model

Similar to the one-factor model, equation 21 with two latent variables X_1 and X_2 was estimated with a Kalman Filter on four CDS premia across 537 days of timeseries. As expected, adding an extra latent variable significantly improved the overall fit of the model with the corresponding R^2 statistic for each of the four maturities at 95%, 89%, 98% and 99% respectively. The parameter estimates are presented in the Table 3

[17] Standard erros are derived from the Fisher Information matrix.

[18] The standard deviation of the latent process is an incredible 0.029 when compared to the values for the standard deviations reported in Table 1 for the four CDS maturities are around 0.0002.

Figure 4: Estimated process for the latent variable X_1.

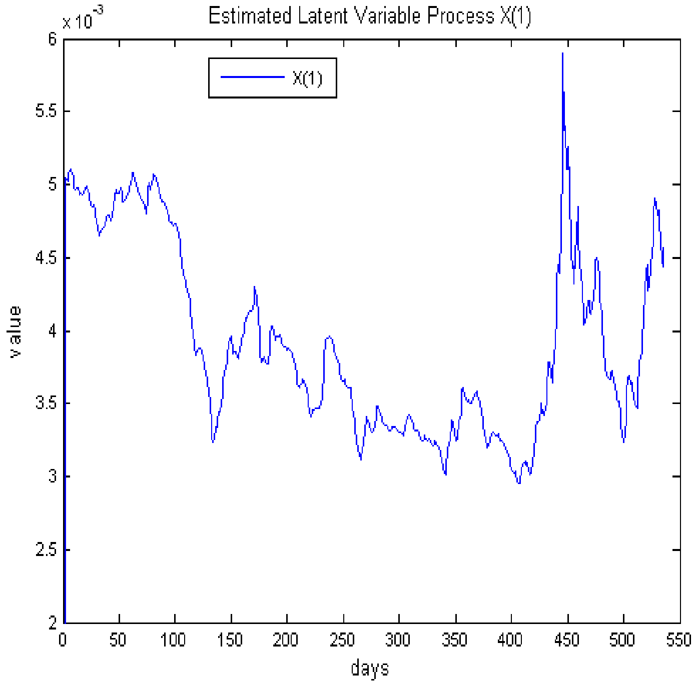
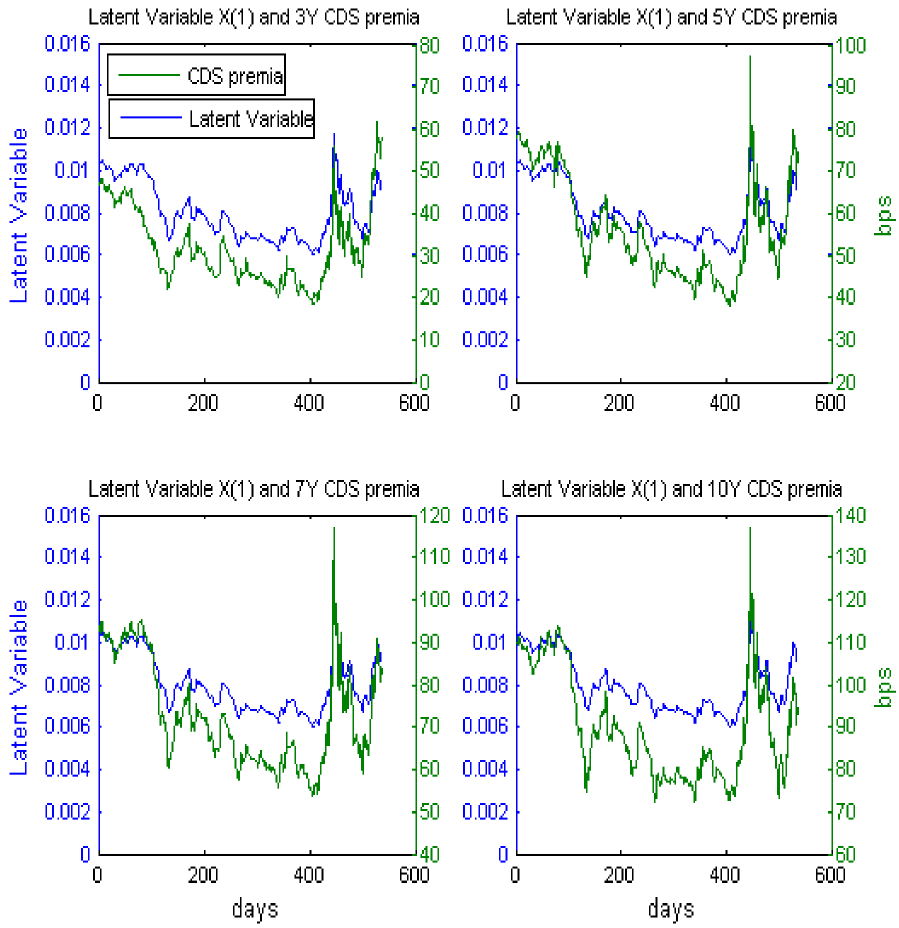


Table 3: Parameter estimates for two-factor model

Parameters	Value
k_1	0.004 $(1.6E-5)$
k_2	0.014 $(1.2E-5)$
θ_1	0.043 $(1.1E-5)$
θ_2	0.011 $(1.0E-5)$
σ_1	0.502 $(1.9E-5)$
σ_2	0.092 $(5.9E-6)$
q_1	-0.012 $(7.6E-6)$
q_2	-0.454 $(6.0E-6)$

Figure 5: Comparison of the estimated latent variable process with the four CDS premia.

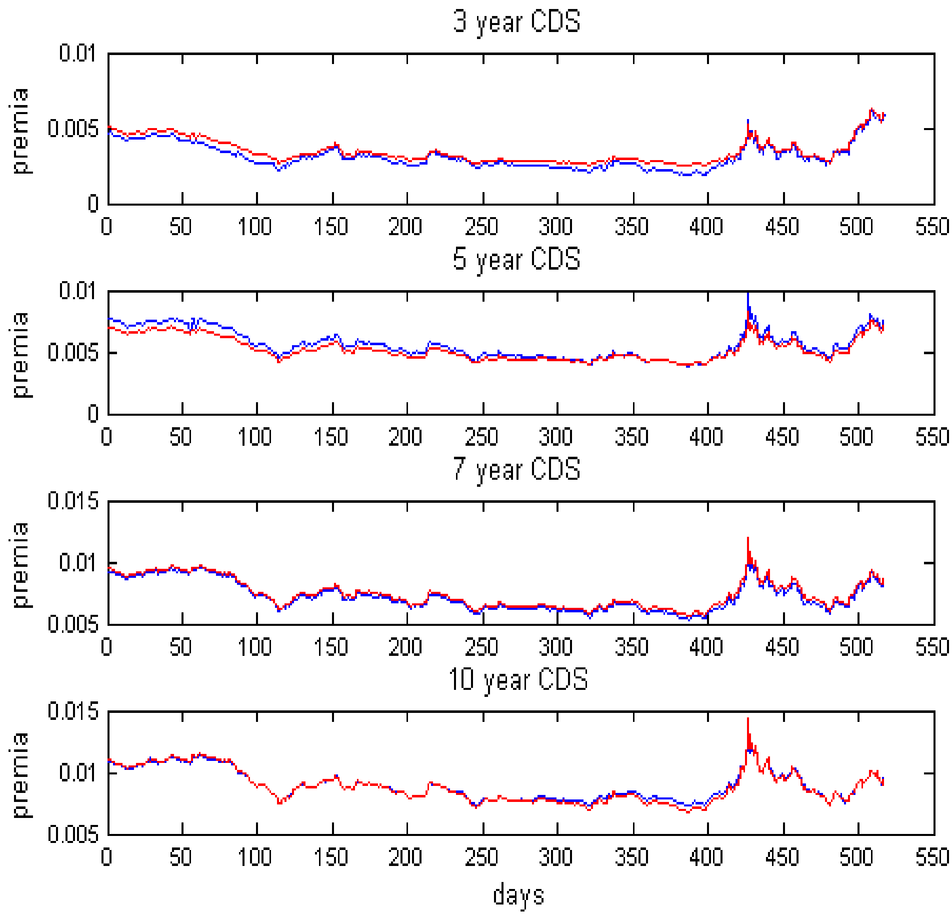


Focusing on the estimates of the parameters. k_1 and k_2 are very close to zero implying slow mean reversion of latent variables to their long-term means θ_1 and θ_2 while exhibiting significant volatility, with σ_1 and σ_2 well above the volatilities observed for CDS premia, as in the one-factor model.

Figure 6 illustrates the fit of the model at various maturities across the whole timeseries. It is interesting to note that the improvement in fit is significant over the one-factor model, especially at shorter 3 and 5 year maturities as well as at the end of the sample where greater turbulence can be observed.

When estimating a model with two variables a modest difficulty arises with the random part of the discretized CIR process (F_t in equation 21) not found in the one-factor model. As it has been noted in the introductory section, by construction, the CIR process does not out rule the possibility of getting ambiguous negative variances and this is exactly the case with the two factor model. The estimated processes for the two latent variables are presented in Figure 7.

Figure 6: Four graphs showing the fit of the two-factor model at various terms of CDS premia across the timeseries. The blue line represents the observed values and the red the estimated values.

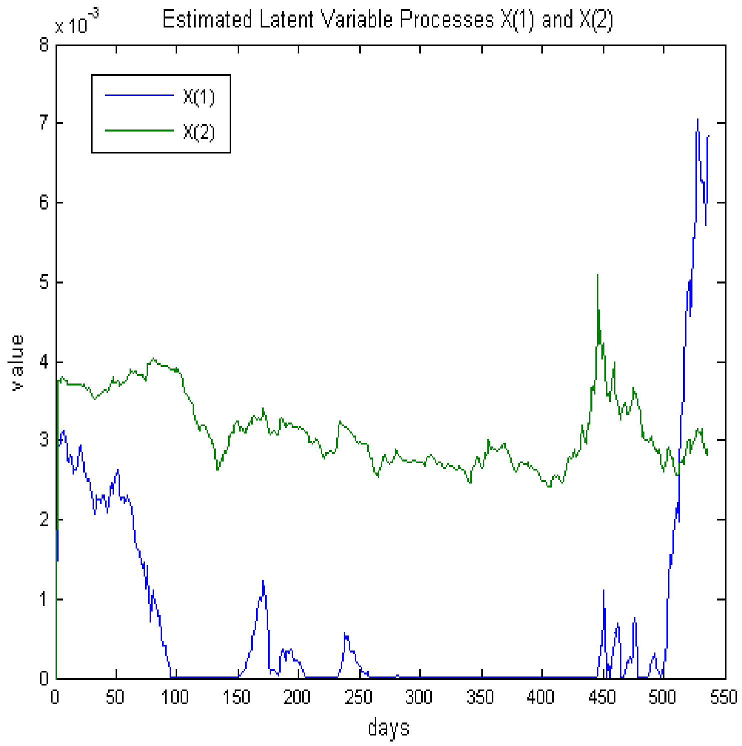


Although ambiguous, a zero value for a latent variable throughout a significant part of the timeseries can be justified as follows. By construction, latent variables are uncorrelated and are driven by some economic forces. Therefore it might occur that the forces driving one of the variables cease to exist, or cancel each other out for a prolonged period of time. The second latent variable seems to come into play with significant and sudden jumps in CDS premia as the ones witnessed in July of 2007 and the following months, in the Figures 7 corresponds to sample number 400 onwards.

In Figure 8 a comparison between the fit of the models to CDS curves is presented. The two graphs have been chosen where the average error of the one-factor model is largest (597.19 basis points) and smallest (7.19 basis points) respectively.

At these two sample dates the improve in fit by adding a second factor into the model is evident. In the first example the average error is reduced from 597.19 bp to 21.75 bp and in second case,

Figure 7: Estimated processes for the latent variables X_1 and X_2. As can bee see the 1st latent variable spent a significant of the sample at its lower barrier fixed at 0.



where the one-factor model performs quite well, the average error is further improved from 7.19 bp to 5.97 bp. Furthermore, by computing a log-likelihood ratio as $2L_2 - 2L_1 = 1902$[19], and comparing it against the critical value of 13.27 taken from the chi-squared distribution with four degrees of freedom (equal to the number of additional parameters in the two-factor model) and percentile of 99%, I conclude that a second factor improves the fit of the model and is necessary to account for the variations at the lower end of the curve.

6.3 Model Implied Probabilities of Default

From the construction of the model in equation 9 it is evident that the probability of default is an essential ingredient to calculating the value of a CDS premia for a given maturity. Furthermore as the probability of default in the estimated model depends solely on a set of latent variables, it becomes theoretically possible to extract the implied risk-neutral probabilities of default from the observed CDS premia as well as compute their evolution.

The average implied risk-neutral probabilities of default for the 3, 5, 7 and 10 year maturities are presented in Table 4

[19] L_1 and L_2 are the log-likelihood functions for the one and two-factor models respectively

Figure 8: Two graphs showing the relative fit of the one-factor model (blue line) and the two-factor model (green line) together with observed values for the CDS curve.

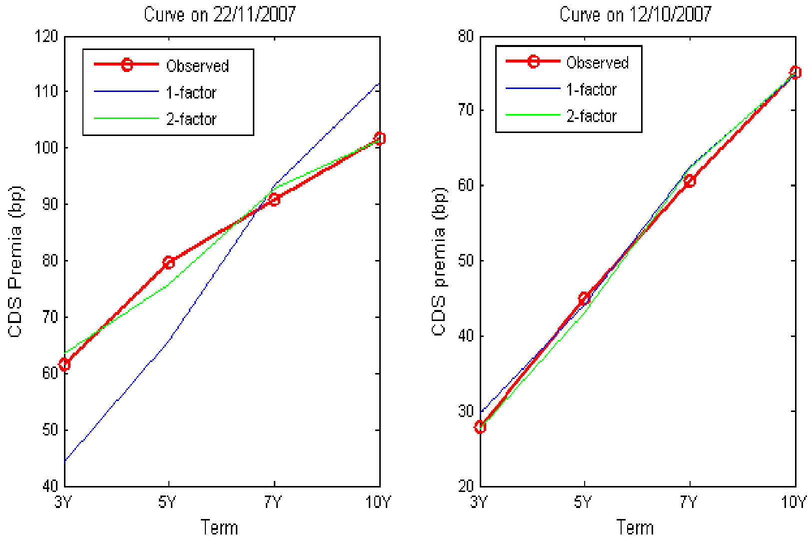


Table 4: Model implied risk-neutral probabilities of default

Term	One-Factor Model	Two-Factor Model
3Y	0.021	0.022
5Y	0.054	0.057
7Y	0.118	0.117
10Y	0.308	0.230

Both one-factor and two-factor models agree quite closely on most of the probabilities, one exception being at the 10-year maturity. I must stress, however, that these probabilities of default are computed from the risk-neutral measure $\mathbb{Q}$ that can differ substantially from the subjective (real world) default probabilities, or the historic default probabilities published by rating agencies. Berndt *et al.* (2005) conducted an extensive study comparing the relation between risk-neutral probabilities implied by models studied in this paper and subjective probabilities calculated by Moody's. Their findings suggest that the ratio between the two can exhibit substantial volatility and rise from 2 at shorter maturities up to 6 for longer maturities.

Results in this paper confirm these large deviations between the two measures, where the ratio of risk-neutral and average historic default probabilities of the entities composing the iTraxx[20] are 2.63, 3.00, 4.18 and 5.48 for the 3, 5, 7 and 10 year maturities respectively.

[20] Historic default probabilities were computed first by calculating the average rating composition of the entities in the iTraxx index (available in Document (n.d.)) and then mapping these ratings to the estimated defaut probability from the sample 1983-2007 (available in Comment (2008)).

7 Conclusions and Directions for Future Research

In the first few sections of this paper I presented a concise review of the two modeling approaches most widely used in credit risk modeling: Structural models, directly linking the probabilities of a firm's default to its internal structure, and more recent Intensity models, to which a firm's internal structure is of no importance. I further concluded that, in a market with imperfect information structural models are a special case of broader Intensity models.

A general Intensity model was parametrized with an affine intensity, where the intensity is a function of a set of latent variables following uncorrelated CIR diffusion processes. This model was then adapted to fit the CDS payment structure at four maturities and two sets of models were estimated with a Kalman Filter approach on timeseries comprising 537 day observations. A model with one latent variable for the intensity process demonstrated encouraging results for the three longest maturities, however lacking in fit for the shortest, 3 year maturity. By adding an additional latent variable it significantly improved the fit of the model to account for 95%, 89%, 98% and 99% of variations at the lowest to highest maturities respectively. A number of difficulties were encountered during the estimation, namely a linearized version of the model had to be used due to the Kalman Filter's inability to handle non-linearity. Additionally in a two-factor model a lower bound of zero had to be imposed on the second latent variable in order to avoid ambiguous negative volatilities.

Further research could extend the results in this paper mainly in two ways. Firstly, by studying the relation between estimated latent and macroeconomic / financial variables in an attempt to establish a link between the CDS curve and real variables. Secondly, to use the model implied default probabilities together with an external source of subjective probabilities to infer on the evolution of risk premia.

8 Appendices

8.1 Feynman-Kac Approach

Feynman-Kac formula tells us that the expected value of an exponential function can be computed as a solution to a partial differential equation.

Theorem 1 [21] *The Feynman-Kac Formula. Let* $(f, R) \in \mathbb{R}^n \times \mathbb{R}^n$ *and*

$$f(X_t, T-t) = E_t\left[e^{-\int_t^T R(X_s)ds} f(X_T, 0)\right] \tag{23}$$

[21] For a technically complete version of the theorem refer to Oksendal (1998)

then

$$\begin{aligned}\frac{\partial f}{\partial (T-t)} &= Af - Rf \\ f(x,0) &= f(X_\tau, 0) \text{ (boundary condition)}\end{aligned} \tag{24}$$

moreover if a bounded function $w(x, T-t)$ solves above differential equation and its associated boundary condition, then it is an admissible solution $f(x, T-t) = w(x, T-t)$. The parameter Af is also said to be an infinitesimal generator of the function f.

In general the solution to the partial differential equation is not trivial albeit in a few special cases. Affine specification of parameters is one of these special cases for which a solution to the PDE is known in closed form. For an intuitive definition of a generator it helps to imagine some function $f(X_t, t)$ that depends on the evolution of some stochastic process X_t, now if we apply Ito's lemma to calculate $\frac{df(X_t,t)}{dt}$ we should come to a result who's first component is deterministic and the second component is purely random. For an infinitesimal interval of time, an infinitesimal generator is the part of this resulting function that if removed leaves us with a martingale. An infinitesimal generator of a Lévy process in $\mathbb{R}^n$ can be found in Duffie *et al.* (2000) and is outlined in the following Proposition

Proposition 2 *For a Lévy process of the form $dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t + \Psi dZ_t$ with vector dimensions $\mu : D \to \mathbb{R}^n$, $\sigma : D \to \mathbb{R}^n \times \mathbb{R}^n$, jump probability λ and a fixed jump size distribution Ψ on $\mathbb{R}^n$ governing the jump sizes z. An infinitesimal generator of a function $f(X_t, t)$ takes the form*

$$Af = \mu(x)\frac{\partial f}{\partial x} + \frac{1}{2}\sum_{i=1}^{n}\left[\sigma(x)'\sigma(x)\frac{\partial f^2}{\partial x \partial x}\right]_i + \lambda(x)\int_{\mathbb{R}^n}[f(x+z) - f(x)]\,d\Psi(z) \tag{25}$$

If all parameters in the model are of affine dependence such as

$$\mu(X_t) = \mu_0 + \mu_1 X_t \tag{26}$$

$$\sigma(X_t) = \sigma_0 + \sigma_1 X_t \tag{27}$$

$$R(X_t) = \rho_0 + \rho_1 X_t \tag{28}$$

$$\lambda(X_t) = l_0 + l_1 X_t \tag{29}$$

then the infinitesimal generator is also of affine form and thus the Feynman-Kac formula can be applied to obtain a partial differential equation. Furthermore as described in Duffie (2005) and Piazzesi (2004) the resulting partial differential equation can be reduced to two ordinary differential equations if the solution of the term structure model takes the form

$$f(X_t, T-t) = e^{\alpha(T-t)+\beta(T-t)X_t} \tag{30}$$

where $\alpha(\tau)$ and $\beta(\tau)$ for $\tau = T-t$ solve the following differential equations

$$\frac{d\alpha(\tau)}{d\tau} = \rho_0 - \mu_0\beta(\tau) - \frac{1}{2}\beta(\tau)'\sigma_0\beta(\tau) - l_0(\theta(\beta(\tau)) - 1) \tag{31}$$

$$\frac{d\beta(\tau)}{d\tau} = \rho_1 - \mu_1\beta(\tau) - \frac{1}{2}\beta(\tau)'\sigma_1\beta(\tau) - l_1(\theta(\beta(\tau)) - 1) \tag{32}$$

where $\theta(\beta(\tau)) = \int_{\mathbb{R}^n} e^{\beta(\tau)z} d\Psi(z)$ determines the jump-size distribution

8.2 Kalman Filter Optimization Algorithm

The Kalman Filter optimization algorithm is constructed from two blocks:

- A Non-linear Kalman Filter for computing estimates of the latent variables at each time period of the sample with a given set of parameters in the transition and measurement equation;
- A quasi-Newton optimization routine maximizing the maximum likelihood function and adjusting the parameter set in the transition and measurement equation.

The two blocks above are executed iteratively until an optimum is reached.

8.2.1 Kalman Filter Block

For a two-dimensional case with a model composed of two latent variables the measurement and transition equations are

$$X_t = \theta\left(1 - e^{-k}\right) + e^{-k}X_{t-1} + F_t \tag{33}$$

$$c_t = H(X_t) + R_t \tag{34}$$

Given that matrix $H(X_t)$ in the measurement equation is a non-linear function of latent variables it must first be linearized. Linearization is done by differentiating the matrix with respect to each of the X_t thus constructing to what normally is referred to as the Jacobian matrix that I will denote $\not{H}_t$. With the linearized version of the model and initial algorithm parameters $\hat{X}_0 = 0$ and $P_0^- = 0$ the following four steps are executed iteratively for every observation in the time-series

1 - Compute optimal Kalman *Gain*

$$K_t = P_t^- \not{H}_t^{'} \left(\not{H}_t P_t^- \not{H}_t^{'} + R_t \right)^{-1} \tag{35}$$

2 - Update latent variable estimates and compute the filter error term

$$\hat{X}_t = \hat{X}_t^- + K_t \left(c_t - \not{H} \hat{X}_t^- \right) \tag{36}$$

$$v_t = c_t - \not{H} \hat{X}_t \tag{37}$$

3 - Update estimate covariance P_t. Calculate the filter error covariance Σ_t and calculate the maximum likelihood value L_t based on the assumption that filter errors are normally distributed

$$P_t = \left(I - K_t \not{H}_t \right) P_t^- \tag{38}$$

$$\Sigma_t = \not{H}_t P_t \not{H}_t^{'} + R_t \tag{39}$$

$$L_t = \ln \left[\det \left(\Sigma_t \right) \right] + v_t' \Sigma_t v_t \tag{40}$$

4 - Calculate optimal predictions for the covariance matrix, latent variables and recalculate the linearization of matrix H_t

$$P_{t+1}^- = A P_t A^{'} + F_t \tag{41}$$

$$\hat{X}_{t+1}^- = A \hat{X}_t \tag{42}$$

$$\not{H}_{t+1}^- = Jacobian \left(\hat{X}_{t+1}^- \right) \tag{43}$$

Restart at n^o 1 with $P_t^- = P_{t+1}^-$, $\hat{X}_t^- = \hat{X}_{t+1}^-$and $\not{H}_t = \not{H}_{t+1}^-$

At the end of the iteration, when the last data point of the sample is reached, the maximum likelihood function $\not{L}$ is calculated as the sum of individual likelihood values

$$\not{L} = \sum_{t=1}^{T} L_t \tag{44}$$

This value is then passed onto the Optimization block discussed in the next subsection.

8.2.2 Optimization Block

For the optimization algorithm I used a preprogrammed matlab function *fmincon*, details on which can be found in online documentation at: http://www.mathworks.com. From online documentation:

> fmincon uses a sequential quadratic programming (SQP) method. In this method, the function solves a quadratic programming (QP) subproblem at each iteration. An estimate of the Hessian of the Lagrangian is updated at each iteration using the BFGS formula.

After a new set of parameters is chosen they are fed back into the Kalman Filter Block where the maximum likelihood is calculated and sent back to the Optimization Block. This process is repeated until no further improvement in fit is attainable.

8.3 Kalman Filter Linearization Methods

Consider the Kalman Filter measurement equation with a non-linear matrix $H(X_t)$ in X_t

$$c_t = H(X_t) + R_t \tag{45}$$

Being a Kalman Filter linear by construction, in its original form, it has not been made to handle non-linear models proposed here. However a non-linear version of the filter, the Extended Kalman Filter can be applied by linearizing the $H(X_t)$ through calculation of its first derivatives with respect to X_t (commonly known as the Jacobian). There is, however, a small detail to the linearization technique, as it must be chosen around which values the derivatives should be evaluated. Apparently there are three alternatives with different levels of complexity and computer time execution. In this paper I make use of the third alternative that demonstrated fair trade-off between the speed of execution and parameter estimates.

The first one (at times referred to as the Extended Kalman Filter), being also the most computer time consuming, is to calculate the Jacobian at each step and evaluate around the latest estimates for X_t. This can be problematic during the first steps while the filter is still converging to its steady state and divergence can arise if initial values were provided inaccurately, or if there is a sudden rise in volatility of the observable variables.

The second one, is as time consuming as the first one, but that does not tend to diverge as often. Here the Jacobian is calculated at each step, but evaluated around the latent variable long-run mean θ. The possibility of divergence is somewhat decreased due to the fact that the long-run mean of the process remains constant at each step, thus reducing the volatility of the Jacobian values at each step.

The third one (at times referred to as the Linearized Kalman Filter), and the least time consuming, is to calculate the Jacobian only once around the long-run mean θ at the beginning of the filter. With this, time consumption can be reduced by the order of thousands. Assuming that in the first two alternatives the filter requires around 500 iterations to converge and takes 10 seconds for each iteration, then with this alternative the number of iterations can be decreased to around 200 and the time to around 1 second. The main disadvantage is that it can lead to poor estimates of the Jacobian matrix and in turn poor estimates of the path for the latent variable.

From the three alternatives described above, the first one seems to be the most commonly used. Duffee (1999) and Chen & Scott (2003) made use of the first alternative in their study on interest rate yield-curves. The third alternative is regularly published in technical books on engineering. I did not find any study confirming the use of the second alternative.

8.3.1 Negativity Problem in Kalman Filter

By choosing the CIR process as the underlying process for latent variables, one is inevitably confronted with a problem of negative variance[22] in the Kalman filter transition equation (equation 22). Negative volatility, additionally causes the maximum likelihood to go into the complex domain. There are three solutions to this problem, two theoretical and the third one purely technical.

The first solution to the negativity problem and the one employed by Duffee (1999) and discussed as viable in Chen & Scott (2003) is to simply reset all the negative values of latent variables to zero when they arise. This causes zero to function as a reflection barrier, however in practical terms it happens so that zero works as an absorber for the latent variable, when reached, the latent variable can stay there for a significant period of time. A possible interpretation is that the forces responsible for movements in the latent variable either cancel each other, or cease to exist for a certain period of time.

The second solution, as discussed in the original paper on CIR processes by Cox *et al.* (1985) is to impose a Kuhn-Tucker type restriction on parameters. The authors argue that by imposing $2k\theta \geq \sigma^2$ on each latent variable process, the upward drift is sufficiently large to make the origin inacessible, in other words, an initial non-negative value for the latent variable can never subsequently become negative. In practice, however, a number of studies[23] suggest that this condition

[22] By construction, in the CIR diffusion process the value of the latent variable affects the volatility through a square-root, thus negative values for the latent variable inevitably cause the process to go into the complex domain.

[23] Studies with models including more than one latent variable tend to indicate relatively low values for the parameters k and θ for at least one of the latent variables, many times violating the Kuhn-Tucker condition of $2k\theta \geq \sigma^2$. Such studies include those of Chen & Scott (2003), Duan & Simonato (1999) and Duffee (1999)

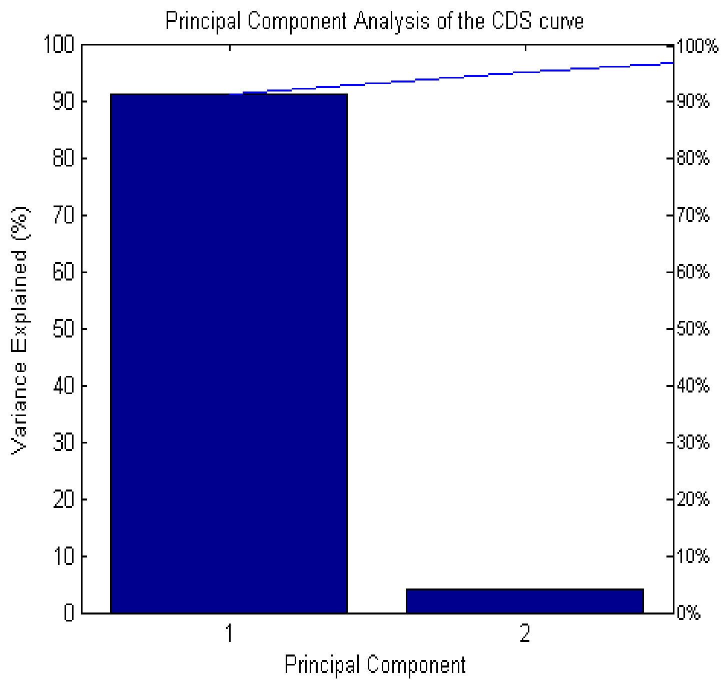
is rarely satisfied when confronted *a posteriori* due to low values of k and θ for at least one latent variable in a multi-factor model.

A third, purely technical solution is for the numerical optimization routine used in the filter to cancel optimization when a negative value is encountered, assume the parameter as not viable and retry with a new set. This solution can potentially have a drastic effects on the optimization routine. The optimization routine is constructed to receive a feedback from the estimation routine for every choice of parameters and the value of this feedback determines the next choice of parameter values. However if the feedback is returned, for example, as a binary value, then there is no way to determine by how much the previous values should be adjusted in order to improve the fit of the model.

8.4 Principal Component Analysis of the CDS Curve

Principal components and their contribution to the explanatory power of the CDS curve were estimated using the matlab *princomp* function. For details on the workings of this function refer to the product documentation available at: http://www.mathworks.com

Figure 9: The bars on the graph represent the total contribution of each principal component to the explanatory power of the variances in the iTraxx Hi-Vol CDS curve. Vertical axes on the right shows the total variance explaned by both components.



Matlab code for calculating principal components and plotting the graph in Figure 9 is presented below

```
%% Clear all variables from memory
clear all
clc
%% Load data
% Structure:  Time series in lines - Maturities in columns
load('itraxx.mat');
%% Calculate Principal components
[pcs,newdata,variances,t2] = princomp(diff(itraxx./10000)); % Compute principal
components on 1st diferences
percent_explained = 100*variances/sum(variances); % Percentage of variance explained
by each component
cumulative_explained = cumsum(percent_explained) % Cumulative variance explained
by each component
%% Plot Results
figure, pareto(percent_explained)
xlabel('Principal Component')
ylabel('Variance Explained (%)')
```

8.5 Estimation Program

Here I will provide a rough outline of the scripts used in estimating the one-factor model. The two-factor model shares the same set of equations, only differing in the number of variables. The program is composed of 3 matlab scripts that interact with each other during estimation, and 2 matrices containing data with CDS premia and risk-free interest rates.

8.5.1 Script estim.m

From this script the estimation process is launched. Care should be taken to execute the script in blocks, to avoid ambiguous results. The first block loads two matrices into memory and sets initial value for parameters. The second block executes a test run to make sure that all of the information is available. The third block is responsible for the actual optimization. In the fourth and fifth blocks statistics are calculated and figures are plotted.

```
% Code should be run sequentially in blocks
%% 1st block
clear
```

```
warning('off','MATLAB:singularMatrix');
warning('off','MATLAB:nearlySingularMatrix');
global y; %set the CDS premia variables to be available in all scripts
global rc; %set the interest rates to be available in all scripts
load('cdsdata.mat'); %load the initial matrix with CDS premia
y = (cdsdata/10000)'; clear cdsdata %set CDS premia to matrix y
load('rc40.mat'); %load interest rates
p(1) = 5.4481e-007; % k1, speed of convergence (transition equation)
p(2) = 0.015884; % b1, long-run mean (transition equation)
p(3) = 0.029705; % s1, volatility (transition equation)
p(4) = -0.36725; % q1, risk premium (measurement equation)
%% 2nd block - Test Routine
% This block returns
% mlerr - current maximum likelihood
% ky - estimated CDS premia
% kx - estimated latent variable time-series
% kv - model errors
% kxest - estimated (anterior) latent variable time-series
% pdefault - prob.  default implicit in each CDS maturity
[mlerr, ky, kx, kv, kxest, pdefault] = estimateproc(p);
%% 3rd block - Optimization Routine
% This block finds the minimum of the maximum likelihood function
% careful as it is very computer and time intensive.  To stop execution
% press Ctrl-C
% pvar - at the end of execution return the standard errors^2
A = []; b = []; Aeq = []; beq = [];
lb = [0; 0; 0; -1];
ub = [1; 5; 5; 0];
nonlcon = [];
options = optimset('MaxIter',50000,'MaxFunEvals',3000,'DiffMinChange',1e-07,'DiffMaxChange',1e-
% optimization options
[pest,fval,exitflag,output,lambda,grad,hessian] = fmincon(@estimateproc,p,A,b,Aeq,beq,lb,ub,non
options); % optimization
```

```
pvar = inv(hessian); % Var-Covar matrix of the estimates = inv(fisher matrix): hessian
= Fisher Information Matrix
warning('on','MATLAB:nearlySingularMatrix');
warning('on','MATLAB:singularMatrix');
%% 4th block - Figures
% Prints out some graphs
CDStxt = [3;5;7;10];
figure,
for k = 1:4
 subplot(4,1,k)
 plot(y(k,20:length(y))); hold on; plot(ky(k,20:length(ky)),'r'); hold off
 title(['CDS(',num2str(CDStxt(k)),') actual(blue) vs. estimated(red)'])
end
clear k
clear sse sse R
%% 5th block - R2 Statistic
% Calculates R^2
for k = 1:4
 for i = 1:length(y)
 sse(k,i) = (y(k,i)*10000-ky(k,i)*10000)^2; % Sum of Squared Errors
 sst(k,i) = (y(k,i)*10000-mean(y(k,:))*10000)^2; % Total Sum of Squares
 end
 R(k) = sum(1-sse(k,:)/sst(k,:)); % R^2
end
```

8.5.2 Script estimateproc.m

This script can only be executed from within *estim.m* and is responsible for executing the Kalman Filter routine and calculating the maximum likelihood value. Due to the non-linearity of model there is a third script (*calcjacobian.m*) that is called to calculate the Jacobian matrix of the measurement equation 21.

```
function [mlerr, ky, kx, kv, kxest, pdefault] = estimateproc(p)
tic
global y;
global rc;
```

```
nsimul = length(y);
% Set of parameters affecting both
c = 0;
k1 = p(1);
b1 = p(2);
s1 = p(3);
q1 = p(4);
dt = 1/360; % Actual days in a year
T = [3 5 7 10]; % Array of CDS maturities
l = 0.40; % Loss Given Default (LGD)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Transition equation variables
A = [exp(-k1*dt)];
d = [b1*(1-exp(-k1*dt))];
R = [1e-7 0 0 0; 0 1e-7 0 0; 0 0 1e-7 0;0 0 0 1e-7];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% kalman initial variables
% The purpose of these variable is to allocate memory and make sure that no
% operation is done on a non existent variable
%kxmin = [0.1; 0.1];
kxmin = 0.01; % anterior latent variable estimates (initial state)
kx = 0; % latent variable estimates
ky = y(:,1); % observed CDS premia
Pmin = s1^2; % anterior kalman P matrix
P = 0; % kalman P marix
I = 1; % identity marix for the univariate case
kv = zeros(4,1); % CDS estimation error
pdefault = 0; % probability of default
kxest = 0; % latent variables estimated values
sigmaM = 0; % sigma matrix
L = zeros(1,length(y)); % log-likelihood
broke = 0; % control variable.  When =1 --> the process broke due to
convtol = 1; % control variable
pdef = 0; % default value of probability of default
```

```
%% Simple 1 time linearization around [b1 b2]
%[kyhat Hj] = calcjacobian(T,[k1 k2],c,l,[b1 b2],[b1; b2],i);
[kyhat Hj pdef] = calcjacobian(T,k1,c,l,b1,b1,q1,s1,1,3); % only spits out Hj with
kyhat = 0
%% Kalman loop
for i = 1:nsimul
 % Jump through the first iteration in order to execute the next
 % iteration with a full set of parameters
 if i <= convtol
 continue
 end

 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 % Calculate Estimation Error %
 kv = y(:,i) - Hj*kxmin;

 %%%%%%%%%%%%%%%%%%%
 % Calculate Sigma %
 % Sigma is used as input parameter in the 1st step and in the maximum
 % log-likelihood function
 sigmaM = Hj*Pmin*Hj'+R; %if det(sigmaM) < 0; dbstop in estimateproc.m at 46; end
 invsigmaM = inv(sigmaM);

 L(i) = log(det(sigmaM)) + kv'*invsigmaM*kv; if i == 2; L(i-1) = L(i); end
 if ~isreal(L(i)); broke = 1; break; end % reset complex values to 0

 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 % Calculate Var-Covar Matrix %
 % This is the var-covar marix of the Transition equation
 Q = [((s1^2)*(kx(1)*(exp(-k1*dt)-exp(-2*k1*dt))+(b1/2)*(1-exp(-k1*dt))^2))/k1];

 %%%%%%%%%%%%
 % 1st step %
 %
```

```
K = Pmin*Hj'*invsigmaM;

%%%%%%%%%%%%
% 2nd step %
% two alternatives can be used:  either use an aproximate value for
% 'kyhat' as Hj*kxmin; or recalculate kyhat each time which consumes
% quite a lot of time
%kx = kxmin + K*(y(:,i) - kyhat);
kx = kxmin + K*(y(:,i) - Hj*kxmin);
if kx(1) < 0; kx(1) = 0; end % reset to 0 negative kx

%%%%%%%%%%%%
% 3rd step %
%
P = (I - K*Hj)*Pmin;

%%%%%%%%%%%%
% 4th step %
%
kxmin = d + A*kx;
Pmin = A*P*A' + Q;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Calculate output parameters %
% kyhat --> estimated CDS spread
% pdefault --> estimated probability of default at each maturity
% kxest --> estimated latent variable
%[kyhat Hj pdef] = calcjacobian(T,k1,c,l,b1,kx,q1,s1,i,3); %Comment during optimization.
Jacobian around current latent variable value
%[kyhat Hj] = calcjacobian(T,k1,c,l,b1,b1,q1,s1,1,2); % Jacobian & kyhat around
b1 --> long-term trend

%%%%%%%%%%%%%%%%%%%%%%%%
% Predicted CDS premia %
```

```
 ky(:,i) = Hj*kxmin;

 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 % default probabilities and latent variable estimates %
 if i == 2; kxest = 0; pdefault = pdef; end
 kxest(i) = kxmin;
 pdefault(i,:)  = pdef;

end
clc % clear output screen
p % print current parameter values
%%%%%%%%%%%%%
% Calculate the sum of individual log-likelihoods.  The model is optimized
% at the mimium of this value.  At times a problem arises when the
% volatility of the transition equation goes below zero, in this case the
% likelihood function is reset to 0
if broke == 0
 mlerr = sum(L(convtol:end))
else
 mlerr = 1/eps
end
toc
```

8.5.3 Script calcjacobain.m

This script is responsible for calculating the Jacobian matrix around the values provided by *estimproc.m*. Note that this script is highly illegible due to its highly optimized structure for velocity of execution.

```
function [kyhat Hj pdef] = calcjacobian(T,k,c,l,b,x,q,s,i,inputs)
%%%%%%%%%%%%%%%%%%%%%%%%%
% Definition of "inputs" variable
% 1 - kyhat
% 2 - Hj
% 3 - kyhat & Hj & pdef
%
```

```
% Highly optimized routine made illegible in order to save around 1000% of
% time taken
ct = 0.25; % coupon yearly periodicity
global rc;
% prealocate space for matrices
kyhat = [];
Hj = [];
pdef = [];
r = rc(i,:)/360; % the affine constant can enter into the model by affecting directly
the interest rate
th1 = sqrt((k+q)^2+2*s^2);
thnom1 = sqrt((k+q)^2+2*l*s^2); % LGD (l) enters through here
d1 = @(t)[b*(1-exp(-k*t))];

A1 = @(t)[((2*th1*exp(0.5*(k+q-th1)*t))/(2*th1*exp(-th1*t)+(k+q+th1)*(1-exp(-th1*t))))^((2*k*b)
B1 = @(t)[(2*(1-exp(-th1*t)))/(2*th1*exp(-th1*t)+(k+q+th1)*(1-exp(-th1*t)))];
Anom1 = @(t)[((2*thnom1*exp(0.5*(k+q-thnom1)*t))/(2*thnom1*exp(-thnom1*t)+(k+q+thnom1)*(1-exp(-
Bnom1 = @(t)[(2*(1-exp(-thnom1*t)))/(2*thnom1*exp(-thnom1*t)+(k+q+thnom1)*(1-exp(-thnom1*t)))];

ri = @(t)[exp(-r(t/ct)*t)];
ni = @(t)[exp(-c*t)*A1(t)*exp(-B1(t)*x(1))];
niT = @(t)[exp(-c*t)*Anom1(t)*exp(-Bnom1(t)*x(1))];
for v = 1:length(T)
 %% Calculate KY
 tau = T(v);
 clear j
 if v > 1
 denominator(v) = denominator(v-1);
 if inputs >= 2
 Js(v) = Js(v-1);
 Jd1(v) = Jd1(v-1);
 end
 fromj = T(v-1)/ct + 1;
 else
```

```
denominator(v) = 0;
if inputs >= 2
Js(v) = 0;
Jd1(v) = 0;
end
fromj = 1;
end

for j = fromj:tau/ct
speed1 = ri(j*ct)*ni(j*ct);
denominator(v) = denominator(v) + speed1;
if inputs >= 2
Js(v) = Js(v) + speed1;
Jd1(v) = Jd1(v) + B1(j*ct)*speed1;
end
end
nominator(v) = ri(tau)*(1-niT(tau));
if inputs == 1 || inputs == 3; kyhat(v,1) = nominator(v)/denominator(v); end
if inputs >= 2
if inputs == 3; pdef(v) = (1-ni(tau)); end
Hj(v,1) = (Bnom1(tau)*ri(tau)*niT(tau)*Js(v) - (nominator(v))*(-Jd1(v)))/(Js(v)^2);
end
end
```

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